Maximal feedback linearization with internal stability: classification of 2-DOF underactuated mechanical systems

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The under actuation in robotics

A humanoid robot could be in under actuated phase because the length of its feet is limited.

\[ \tau_{ZMP} = Mg(x - p_x) - m\ddot{z}_c = 0 \]
Zeros dynamics equations: A link with the linear case

- The unit step response of the linear system

\[
\frac{Y(s)}{U(s)} = \frac{s - 2}{s - 1}
\]

\[
\frac{s - 2}{s(s - 1)} = \frac{s - 1}{s(s - 1)} - \frac{1}{s(s - 1)} = \frac{s - 1}{s(s - 1)} + \frac{1}{s} - \frac{1}{s - 1}
\]

→ in temporal

\[
y(t) = 2 - e^t
\]

- The unit step response of the corrected linear system

\[
\frac{s - 2}{s} = 1 - \frac{2}{s}
\]

→ in temporal

\[
y(t) = \delta(t) - 2
\]
The critical choice of the output

With the choice of \( \frac{Y(s)}{U(s)} = \frac{s - 2}{s - 1} \) we have:

- a zero in the half right part of the complex plane \( \rightarrow \) system with non-minimum phase
- a pole in the half left part of the complex plane \( \rightarrow \) unstable system
- Possibility of the presence of singularities
Two-degree-of-freedom spring-mass system:

\[ m_2 \]
\[ k \]
\[ m_1 \]
\[ F \]
\[ q_2(t) \]
\[ q_1(t) \]

The expressions for the kinetic energy \( T \) and the potential energy \( V \) are the following:

\[
2T = m_1 \dot{q}_1^2 + m_2 (\dot{q}_1 - \dot{q}_2)^2, \quad 2V = kq_2^2. \tag{5}
\]
This two-degree-of-freedom translational mechanical system is a differentially flat system since the output $y = q_1 - q_2$ has a relative degree 4 and

- $y = q_1 - q_2$
- $\dot{y} = \dot{q}_1 - \dot{q}_2$
- $\ddot{y} = -k(\frac{2}{m_1} + \frac{1}{m_2})q_2$
- $y^{(3)} = -k(\frac{2}{m_1} + \frac{1}{m_2})\dot{q}_2$

define the linearizing coordinates. The state feedback is computed when solving the following equation in $F$

$$v = -k\left(\frac{2}{m_1} + \frac{1}{m_2}\right)\left[k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)q_2 + \frac{F}{m_1}\right]$$

which yields $y^{(4)} = v$. 

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Relative degree: Consider a single-input nonlinear system

\[ \Sigma: \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \]  

where the state \( x \in \mathbb{R}^n \). The relative degree of \( y = h(x) \)
equals \( r \), then locally there exists a regular static state feedback
\( u = \alpha(x) + \beta(x)v \)
and a state transformation
\( (z, w) = \phi(x) \) such that system \( \Sigma \) reads,

\[ \begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
& \quad \vdots \\
\dot{z}_r &= v \\
\dot{w} &= \eta(z, w) \\
y &= z_1.
\end{align*} \]
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subspace $\{\mathcal{H}_k\}$ of $\mathcal{E}$

$\mathcal{H}_0 = \text{span}_K \{dx, du\}$
$\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, \ k \geq 1. \quad (9)$

$\omega = a \, dx + \sum_{k \geq 0} b_k \, du^{(k)} \quad (10)$

$\dot{\omega} = (\dot{a} \, dx + a \, \dot{d}x) + \sum_{k \geq 0} (\dot{b}_k \, du^{(k)} + b_k \, du^{(k+1)}) \quad (11)$

$\mathcal{H}_k$: the set of all one-forms with relative degree at least $k$. $\mathcal{H}_1 = \text{span}_K \{dx\}$; $\mathcal{H}_2 = \text{span}_K \{g\}^\perp$. 
The zero dynamics of system Σ, given by (7):

\[ \dot{w} = \eta(0, w) \]

internal dynamics consistent with the constraint \( y(t) \equiv 0 \).

\[ y = q_i - q_i^d = 0, \quad \dot{y} = \dot{q}_i - \dot{q}_i^d = 0 \]
\[ \ddot{y} = \ddot{q}_i - \ddot{q}_i^d = 0 \]

from the angular momentum theorem → \( f(\theta, \dot{\theta}, \ddot{\theta}) = 0 \),
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Definition 0.1 The Lagrangian \( \mathcal{L} \) is defined as
\[
\mathcal{L}(q, \dot{q}) := T(q, \dot{q}) - V(q),
\]
where \( T \) denotes the kinetic energy and \( V \) is the potential energy. Lagrange’s equations of motion for a mechanical system are:

\[
\frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \mathcal{L}(q, \dot{q}) - \frac{\partial}{\partial q_i} \mathcal{L}(q, \dot{q}) = \gamma_i
\]  

(12)

\( \gamma_i \) is the external generalized force at the \( i \)-th joint.
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**Definition 0.2** $q_i$ is said to be **cyclic** with respect to a dynamical system, if the following holds [Arnold97]:

$$\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q_i} = 0$$

where $\mathcal{L}(q, \dot{q})$ is the Lagrangian of the system.

**Remark:** Theorem d’Emi Noether (1918): Symmetry of rotation $\Rightarrow$ conservative angular momentum. Direct joint between law of conservation of Euler-Lagrange equations and symmetries.
Property 0.3  The kinetic energy of a mechanical system is invariant under a translation or a rotation of the world frame, see [Spong89]. → for the equations of motion of a two-degrees of freedom mechanical system, with a position variable or an orientation variable with respect to the world frame and a joint variable as components of the generalized vector, the inertia matrix does not depend on the position variable or the angular orientation with respect to the world frame.

\[
\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{D} \dot{\mathbf{q}}.
\]
Generalized conjugate momenta: \( \frac{\partial}{\partial \dot{q}_i} \mathcal{L}(q, \dot{q}) \) for \( i \in [1, \cdots, N] \), where \( N \) is the number of generalized coordinates.

Property 0.4 The generalized conjugate momenta are equal to \( \frac{\partial}{\partial \dot{q}_i} \mathcal{L}(q, \dot{q}) = d_i(q) \dot{q} \) where \( d_i \) is the \( i-th \) row of the inertia matrix \( D \).

From the definition of the generalized conjugate momenta, the Lagrange’s equation (12) can be rewritten as:

\[
\frac{d}{dt} (d_i(q) \dot{q}) = \frac{\partial}{\partial q_i} \mathcal{L}(q, \dot{q}) + \gamma_i
\]  

(13)
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Theorem 0.5

\[ \mathcal{H}_2 = \text{span}_K \{ g \} = \text{span}_K \{ dq_1, dq_2, d(d_i(q)\dot{q}) \} \]

where \( i \) is the index of the unactuated variable. For a two-degree of freedom underactuated mechanical system, if \( q_1 \) is the actuated variable (respectively \( q_2 \)), the generalized conjugate momentum

\[ \frac{\partial}{\partial \dot{q}_2} \mathcal{L}(q, \dot{q}) = d_2(q)\dot{q} \] (respectively

\[ \frac{\partial}{\partial \dot{q}_1} \mathcal{L}(q, \dot{q}) = d_1(q)\dot{q} \] is a function with relative degree at least 2; thus its differential belongs to \( \mathcal{H}_2 \).
Theorem 0.6 For 2-Dof mechanical systems $\mathcal{H}_3$ is at least always partially integrable such that:

$$dp(q) = dq_1 + \frac{d_{i2}}{d_{i1}} dq_2 \in \mathcal{H}_3.$$  

Consider $d_i(q_2) dq = d_{i1}(q_2) dq_1 + d_{i2}(q_2) dq_2$ $i$ denotes again the index of the unactuated variable.

$$dp = dq_1 + \frac{d_{i2}(q_2)}{d_{i1}(q_2)} dq_2.$$  

is an exact one-form. Consequently $dp \in \mathcal{H}_3$ and

$$\mathcal{H}_3 = \text{span}_K \{ \omega, dp \}$$  

for some $\omega$ which may be exact or not.
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Calculation of $\omega$

- $q_1$ is the unactuated variable.

\[ \mathcal{H}_3 = \text{span}_K \{d\rho, d(d_{11}\dot{p})\}. \] (17)

- $q_2$ is the unactuated variable.

**Theorem 0.7** For a two-degree of freedom underactuated mechanical system, $\mathcal{H}_3$ is fully integrable if and only if

\[ |D| = \rho d_{22}, \] where $\rho$ is a non zero real number. Moreover the set $\mathcal{H}_3$ is such that:

\[ \mathcal{H}_3 = \text{span}_K \{d\rho, d \left( \frac{d_{21}^2}{d_{22}} \dot{p} \right) \}. \] (18)
Definition 0.8 A two-degree-of-freedom mechanical system is said to be of Class 0 if it is accessible and $\mathcal{H}_4$ is integrable.

Exemple: Flywheel Pendulum

$q_2$ is cyclic. The inertia wheel pendulum is a differentially flat system since the output $y = p$ has relative degree 4 and
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Define the linearizing coordinates. The state feedback is computed when solving the following equation in $\Gamma$

$$v = \frac{b}{a_{11} + a_{22}} \ddot{q}_1 \sin q_1 + \frac{b}{a_{11} + a_{22}} \left[ \frac{b}{a_{11}} \sin q_1 + \frac{\Gamma}{a_{11}} \right] \cos q_1$$

which yields $y^{(4)} = v$. 

\[
\begin{align*}
y &= q_1 + \frac{a_{22}}{a_{11} + a_{22}} q_2 \\
\dot{y} &= \dot{q}_1 + \frac{a_{22}}{a_{11} + a_{22}} \dot{q}_2 \\
\ddot{y} &= -\frac{b}{a_{11} + a_{22}} \sin q_1 \\
y^{(3)} &= -\frac{b}{a_{11} + a_{22}} \dot{q}_1 \cos q_1
\end{align*}
\]
Lemma 0.9 Consider a 2-DOF under actuated mechanical system. The system is a class 0 system if its inertia matrix does not depend on the configuration variable.

Let $q_i$ be the unactuated variable $\Rightarrow d_i(q)\dot{q} = \frac{\partial}{\partial \dot{q}_i} L(q, \dot{q})$ has a relative degree 2 at least. If the variable of configuration is cyclic then $d_i(q)\dot{q} = d_i\dot{q}$. From the Lagrange’s equations (12):

$$
\frac{d}{dt} d_i\dot{q} = \frac{\partial}{\partial q_i} (T(\dot{q}) - V(q)) \\
= \frac{\partial}{\partial q_i} (\frac{1}{2} \dot{q}^t D \dot{q} - V(q)) \\
= -\frac{\partial}{\partial q_i} V(q) 
$$

Then $d_i\dot{q}$ has a relative degree 3 at least $\Rightarrow$ the output $y = d_i\dot{q}$ has a relative degree 4.
**Definition 0.10** A two-degree-of-freedom mechanical system is said to be of Class 1 if it is accessible, $\mathcal{H}_4$ is not integrable and $\mathcal{H}_3$ is fully integrable.

**Exemple:** Translational inverted pendulum
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\begin{align}
2\mathcal{T} &= I\dot{\theta}^2 + m_1l_1 + m_2((\dot{x} - l\dot{\theta})^2 + x^2\dot{\theta}^2), \\
\mathcal{V} &= m_1gl_1 \cos \theta + m_2g(l \cos \theta + x \sin \theta).
\end{align}

\begin{align}
dp &= \frac{\sigma dt}{I + m_2(l^2 + x^2)} \\
&= d\theta - \frac{m_2l}{I + m_2(l^2 + x^2)} dx,
\end{align}

\begin{equation}
\mathcal{H}_3 = \text{span}_K \{d\sigma, dp\},
\end{equation}

Any Class 1 system can be maximally linearized with internal stability. The adequate output: in the form of a linear combination of the angular momentum $\sigma$ and its "integral" $p$. 

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Definition 0.11 A two-degree-of-freedom mechanical system is said to be of Class 2 if it is accessible, $\mathcal{H}_4$ is not integrable and the derived flag of $\mathcal{H}_3$ has dimension 1.

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\[
2\mathcal{T} = (M + m)\ddot{e}^2 + 2ml\dot{e}\dot{\theta} \cos \theta + J\dot{\theta}^2, \\
\mathcal{V} = mg(h + l \cos \theta).
\]  

(24)

\[
\mathcal{H}_2 = \text{span}_K \{g\}^\perp \\
= \text{span}_K \{de, d\theta, d(-ml \cos \theta \dot{e} + J\dot{\theta})\}
\]  

(25)

\[-ml \cos \theta \dot{e} + J\dot{\theta}: \text{ the generalized conjugate momentum } \frac{\partial L}{\partial \dot{\theta}}.
\]

\[
\mathcal{H}_3 = \text{span}_K \{\omega_1, dp_1\},
\]

(26)

where $\omega_1$ is a differential one-form and

\[
dp_1 = \frac{1}{ml \cos \theta} \frac{\partial L}{\partial \theta} = (-de + \frac{J}{mgl \cos \theta} d\theta)
\]

$\mathcal{H}_3$ is not fully integrable and moreover, its derived flag $\mathcal{H}_3^{(1)} = \text{span}_K \{dp_1\}$. One thus concludes that the Cart-pole system is Class 2 system.
### Class 2 systems 1/2

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<tr>
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<th>always integrable: $\mathcal{H}_2 = \text{span}_K {dq_1, dq_2, d(d_i(q)\dot{q})}$ $\mathcal{H}_2 = \text{span}_K {dq_1, dq_2, dp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}_3$</td>
<td>$q_1$ not actuated always integrable: $\mathcal{H}_3 = \text{span}<em>K {dp, d(d</em>{11}\dot{p})}$ $\mathcal{H}_3 = \text{span}<em>K {dp, d(\dot{p} \frac{d^2}{d</em>{22}})}$</td>
</tr>
<tr>
<td>$\mathcal{H}_4$</td>
<td>sufficient condition: $q_2$ is cyclic $\mathcal{H}_4 = \text{span}_K {dp}$</td>
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Conclusion

- To state and to solve the maximal feedback linearization problem with internal stability for a general class of two-degree-of-freedom mechanical systems.
- An efficient way to stabilize non flat systems.
- The generalized conjugate momenta appear to be one key of the problem.
- We can tackle more complex mechanical devices (Paraglider, Humanoid robots with Kajita’s model of the inverted pendulum).